Comp. & Maths. with Appls. Vol. 12A, Nos. 4/5, pp. 477–489, 1986
Printed in Great Britain.

0886-9553/86 $3.00 + .00

THE INFLUENCE OF NONLINEAR CONDUCTION ON SINGULARITY FORMATION IN THE INTENSE PLANE-WAVE, NONLINEAR DIELECTRIC INTERACTION PROBLEM†

FREDERICK BLOOM
Department of Mathematical Sciences, Northern Illinois University, DeKalb, Illinois 60115, U.S.A.

Abstract—The propagation of an intense plane-wave pulse into a nonlinear dielectric half-space is considered; it is assumed that the dielectric is not a perfect nonconductor and that the conduction vector **J** is given by a simple nonlinear Ohm's law. Under these conditions we prove that singularities form in the propagating electromagnetic wave provided the initial gradients of the electromagnetic fields in the wave are positive, and pointwise sufficiently large. Our results point to the fact that nonlinear conduction currents may provide a natural dissipative mechanism in the plane-wave–nonlinear dielectric interaction problem.

1. INTRODUCTION

Let $\Omega \subseteq R^3$ be any bounded, or unbounded, open domain occupied by a nonlinear, isotropic dielectric substance which conforms to the constitutive relations

$$\mathbf{D}(\mathbf{x}, t) = \epsilon(\mathbf{E}(\mathbf{x}, t))\mathbf{E}(\mathbf{x}, t), \quad \mathbf{B}(\mathbf{x}, t) = \mu(\mathbf{H}(\mathbf{x}, t))\mathbf{H}(\mathbf{x}, t) \tag{1.1}$$

for $\mathbf{x} \in \Omega$, where **B** is the magnetic field, **H** the magnetic intensity, **E** the electric field, and **D** the electric induction field defined by $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}(\mathbf{E})$; in this latter relation, ϵ_0 is the permittivity of free space and **P** is the polarization vector. Throughout Ω, for $t > 0$, it is assumed that Maxwell's equations apply, in the standard form

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\mathrm{curl}\ \mathbf{E}, \quad \mathrm{div}\ \mathbf{B} = 0, \\ \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} &= \mathrm{curl}\ \mathbf{H}, \quad \mathrm{div}\ \mathbf{D} = \rho, \end{aligned} \tag{1.2}$$

where $\rho = \rho(\mathbf{x}, t)$ is the free-charge density and **J** is the conduction vector. We are going to assume, in this paper, that $\mu(\mathbf{H}) = \mu_0 = \text{const.}$ where μ_0 is the permeability of free space. Although we do not assume a specific form for the scalar-valued vector constitutive function ϵ in this paper, a typical assumption, which has appeared in some of the nonlinear optics literature[1], is that

$$\epsilon(\mathbf{E}) = \epsilon_0 + \epsilon_2\|\mathbf{E}\|^2, \quad \epsilon_0 > 0, \quad \epsilon_2 > 0. \tag{1.3}$$

Our interest in this paper is in the case where Ω is the half-space $\{\mathbf{x} = (x, y, z) | x > 0\}$; we assume that a linearly polarized plane wave is propagating into Ω so that $\mathbf{E} = (0, E(x, t), 0)$, $\mathbf{D} = (0, D(x, t), 0)$, $\mathbf{B} = (0, 0, B(x, t))$ and $\mathbf{H} = (0, 0, H(x, t))$. With this latter assumption the constitutive relations (1.1) reduce to the scalar equations

$$D(x, t) = \bar{\epsilon}(E(x, t))E(x, t) \equiv \mathcal{D}(E(x, t)); \quad B(x, t) = \mu_0 H(x, t). \tag{1.4}$$

If we set $\bar{\epsilon}(\zeta) = \epsilon((0, \zeta, 0))$, $\forall \zeta \in R^1$ and assume that $(\zeta\epsilon(\zeta))' > 0$, $\forall \zeta \in R^1$, then the first

†Research supported, in part, by NSF Grant MCS 8401308.

relation in (1.4) can be written in the form

$$D(x, t) = \int_0^{E(x,t)} a(\zeta)\, d\zeta, \quad a(\zeta) \equiv (\zeta\hat{\epsilon}(\zeta))' > 0. \tag{1.5}$$

Because this last integral is monotone in E, $\exists\, \mathscr{E}$ such that $E(x, t) = \mathscr{E}(D(x, t))$ and $\mathscr{E}'(D) = 1/a(E) > 0$.

In all previous work[2,3–8] on the problem of plane-wave–dielectric half-space interaction, it has been assumed *a priori* that the conduction vector $\mathbf{J} = \mathbf{0}$. In such a situation, it has been shown[2–8] that singularities form in the solutions of initial-value problems associated with the homogeneous quasilinear hyperbolic equations that result, when (1.2) is combined with our constitutive hypothesis and the assumption of plane-wave propagation, even for initial values of the electromagnetic field that are arbitrarily small, smooth, and compactly supported; examples of results of this kind, which have been obtained recently by this author in [2], will be cited below. However, no nonlinear dielectric is a perfect nonconductor and thus it may be expected that $\mathbf{J} \neq \mathbf{0}$ and that, in fact, $\mathbf{J} = \sigma(\mathbf{E})\mathbf{E}$ where σ is the conductivity; this last relationship is an example of a simple nonlinear Ohm's law. We could just as easily assume that our medium is nonhomogeneous so that $\sigma = \sigma(\mathbf{x}, \mathbf{E})$ but this leads to additional complications that will be dealt with in a future work[9]; for the sake of consistency with our assumption of homogeneity we will also assume that ρ is negligible.

Our basic interest in this paper is to begin to explore the effects, on singularity formation in the wave, which result from dropping the *a priori* assumption that $\mathbf{J} \equiv \mathbf{0}$. In referring to this problem of plane-wave–dielectric half-space interaction, Broer[3] expressed the opinion that "in reality the solution of the physical problem may be univalent throughout. This means that there must be an effect not described by (7) and (8) [our system (2.2) below with $\tilde{\sigma} = 0$] which becomes operative whenever waves steepen. This effect must then put a bound on the steepness of the wave fronts. An analogous situation exists in the theory of non-linear acoustics . . . (where) the neglected effects are viscosity and heat conduction. In the optical case the wave propagation will certainly be rendered univalent by some dissipative process not included in the field equations, which admit an energy equation without dissipation. There is, at present, no theory of this effect." In this paper we will effect a partial answer to the hypothesis of Broer by showing that the inclusion of nonlinear conduction in the model leads to the conclusion that singularities form provided that initial gradients of the electromagnetic fields in the wave are (pointwise) sufficiently large; no such restriction on these initial gradients is needed for singularities to form when $\mathbf{J} \equiv \mathbf{0}$. In a forthcoming work[10] it will be shown that with $\mathbf{J} \neq \mathbf{0}$, the assumption of initial data which is both smooth and sufficiently small, in an appropriate sense, the possesses gradients which are, pointwise sufficiently small, assures the existence of a globally smooth electromagnetic field in the wave and thus precludes the formation of singularities.

2. THE SYSTEM OF EVOLUTION EQUATIONS

In all that follows we will assume that the conduction vector $\mathbf{J}$ is given by the nonlinear Ohm's law $\mathbf{J} = \sigma(\mathbf{E})\mathbf{E}$, with $\sigma(\cdot)$ of class C^1, and that ρ, the free-charge density, is negligible. If we set $\tilde{\sigma}(\zeta) = \sigma((0, \zeta, 0))$, $\zeta \in R^1$, then $\mathbf{J} = (0, \tilde{\sigma}(E(x, t))E(x, t), 0)$, and the equation expressing conservation of charge, namely $\partial\rho/\partial t - \nabla \cdot \mathbf{J} = 0$, is obviously satisfied. Combining our assumption of plane-wave propagation with our constitutive relations we find that Maxwell's equations (1.2) reduce to the scalar quasilinear system

$$\tilde{\sigma}(E)E + \frac{\partial \mathscr{D}(E)}{\partial t} = -\frac{\partial H}{\partial x}; \quad \frac{\partial B}{\partial t} = -\frac{\partial E}{\partial x}. \tag{2.1}$$

However, $\partial\mathscr{D}/\partial E = a(E)$ and $B = \mu_0 H$, so we may rewrite (2.1) in the form

$$\begin{pmatrix} E \\ H \end{pmatrix}_t + \begin{pmatrix} 0 & 1/a(E) \\ 1/\mu_0 & 0 \end{pmatrix} \cdot \begin{pmatrix} E \\ H \end{pmatrix}_x = \begin{pmatrix} -\tilde{\sigma}(E)E/a(E) \\ 0 \end{pmatrix}. \tag{2.2}$$

which is a strictly hyperbolic inhomogeneous quasilinear system whose characteristics are defined by the nonlinear ordinary differential equations $dx/dt = \pm 1/\sqrt{\mu_0 a(E(x, t))}$. Although (2.2) is not in conservation form, we may use the fact that $E(x, t) = \mathcal{E}(D(x, t))$ to rewrite this system as

$$\frac{\partial D}{\partial t} + \frac{1}{\mu_0}\frac{\partial B}{\partial x} = -\Sigma(D), \quad \frac{\partial B}{\partial t} + \mathcal{E}'(D)\frac{\partial D}{\partial x} = 0, \tag{2.3}$$

where $\Sigma(D) = \bar{\sigma}(\mathcal{E}(D))\mathcal{E}(D)$; this latter system is an inhomogeneous hyperbolic conservation law of the form

$$\frac{\partial u_i}{\partial t} + \frac{\partial f_i}{\partial x}(\mathbf{u}) = \mathbf{g}(\mathbf{u})$$

with

$$\mathbf{u} = \begin{pmatrix} D \\ B \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} B/\mu_0 \\ \mathcal{E}(D) \end{pmatrix}, \quad \mathbf{g}(\mathbf{u}) = \begin{pmatrix} -\Sigma(D) \\ 0 \end{pmatrix},$$

and associated characteristics defined by the equations

$$\frac{dx}{dx} = \pm\sqrt{\frac{\mathcal{E}'(D(x, t))}{\mu_0}}.$$

For $\sigma \equiv 0$, $\bar{\sigma} = 0$ and (2.3) reduces to a standard hyperbolic conservation law; it is well known that singularities will develop in solutions of initial-value problems for such systems even if the initial data is small, smooth, and compactly supported (e.g. [11] and [12]). In fact, it was shown by this author in [2] that if $D_0(x) \equiv D(x, 0)$ is periodic on R^1, $B_0(x) \equiv 0$, and $\mathcal{E}''(0) \neq 0$ (so that the problem exhibits genuine nonlinearity) then $\nabla_{(x,t)}D \equiv (D_{,t}, D_{,x})$ must blow up in finite time

$$t_{\max} \simeq \frac{\mu_0}{\max|D_0'(x)|} \times \frac{\sqrt{\mathcal{E}'(0)}}{|\mathcal{E}''(0)|},$$

while if $D_0(x)$, $B_0(x)$ both have compact support in R^1, and are of class C^1, then any C^1 solution of the associated initial-value problem must develop singularities in finite time in the first derivatives if $\mathcal{E}'(\zeta)$ is not constant on any open interval.

For $\sigma \not\equiv 0$ we easily find that (2.3) leads to the nonlinear wave equation

$$\frac{\partial^2 D}{\partial t^2} + \Sigma'(D)\frac{\partial D}{\partial t} = \frac{\partial^2 \mathcal{E}(D)}{\partial x^2}. \tag{2.4}$$

Recently, both Nishida[13] and Slemrod[14] have examined the damped quasilinear system

$$\frac{\partial w}{\partial t} - \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} - \Gamma'(w)\frac{\partial w}{\partial x} = -\alpha v \quad (\alpha > 0), \tag{2.5}$$

which yields the damped nonlinear wave equation

$$\frac{\partial^2 w}{\partial t^2} + \alpha\frac{\partial w}{\partial t} = \frac{\partial^2 \Gamma(w)}{\partial x^2}. \tag{2.6}$$

Clearly the evolution equation (2.4) coincides, in form, with (2.6) iff Σ is a linear function of its argument. Nishida[13] considered the system (2.5), with associated periodic initial data $w(x, 0) = w_0(x)$, $v(x, 0) = v_0(x)$, assumed local hyperbolicity, i.e. $\Gamma'(0) > 0$, and proved an *a priori* estimate which shows that, for as long as a C^1 solution (w, v) in (x, t) exists, $\sup_x |w|$ may be held small (by choosing the L^∞ norms of the initial data $w_0(x)$, $v_0(x)$ sufficiently small) so that $\Gamma' > 0$ for as long as the smooth solution exists. By working with the Riemann invariants

$$r = v + \int_0^w \sqrt{\Gamma'(\zeta)d\zeta}, \quad s = v - \int_0^w \sqrt{\Gamma'(\zeta)d\zeta} \tag{2.7}$$

that are naturally associated with (2.5), and studying the behavior of $\partial r/\partial x$, $\partial s/\partial x$ along the characteristic curves defined by $(dx/dt = \pm\sqrt{\Gamma'(w(x, t))}$, Nishida was able to derive *a priori* estimates for

$$\left|\frac{\partial r}{\partial x}(x, t)\right| \quad \text{and} \quad \left|\frac{\partial s(x, t)}{\partial x}\right|$$

along the characteristics; these estimates were then used, in conjunction with a continuation theorem and a standard local existence theorem for initial-value problems associated with systems of the form (2.5) (e.g. [15]) to show that if $\Gamma''(0) > 0$ and the C^1 norms of $w_0'(x)$, $v_0'(x)$ are sufficiently small, a C^1 [in (x, t)] solution (w, v) of the initial-value problem for (1.1) exists for all $t > 0$. In this paper, however, our interest will not be in establishing the counterpart of Nishida's existence result for (2.5) for the system (2.3), but in establishing a counterpart of the nonexistence result proven by Slemrod for the system (2.5), namely that for r, s as defined by (2.7), $\exists t_\infty < \infty$ such that $|\partial r/\partial x(x, t)| \to \infty$ as $t \to t_\infty$ provided that $\Gamma''(0) > 0$, with $r_0'(x)$ sufficiently large and positive at some x.

In terms of the Riemann invariants r, s introduced in (2.7), the system (2.5) may be written as

$$r' = -\frac{\alpha}{2}(r + s), \quad s` = -\frac{\alpha}{2}(r + s), \tag{2.8}$$

where

$$' = \partial/\partial t - \sqrt{\hat{\Gamma}'(r - s)}\partial/\partial t, \quad ` = \partial/\partial t + \sqrt{\hat{\Gamma}'(r - s)}\partial/\partial x,$$

and $\hat{\Gamma}(r - s) \equiv \Gamma(w(r - s))$. For an appropriately defined pair of Riemann invariants (which we also denote by) r, s we will be able to rewrite our system (2.3) in the form

$$r' = \Phi(r - s), \quad s` = -\Phi(r - s), \tag{2.9}$$

with r', $s`$ suitably defined derivatives of r, s along their respective characteristic curves in the (x, t) plane. Now, (2.9) is very nearly a special case of the following system in Riemann invariant form

$$r' = -\frac{\alpha}{2}(r + s) + \bar{\Phi}(x, t), \quad s` = -\frac{\alpha}{2}(r + s) + \bar{\Phi}(x, t), \tag{2.10}$$

for which Hattori[16] has established finite-time breakdown of C^1 solutions by using an argument due to Rozhdestvenskii[17], which involves showing that characteristics of the same family must cross in finite time. However, central to Hattori's analysis is the derivation and use of the *a priori* estimate

$$|r(x, t)| + |s(s, t)| \le \sup_x |r_0(x)| + \sup_x |s_0(x)| + \left(\frac{4}{\alpha}\right) \sup_x |\bar{\Phi}(x, t)|, \tag{2.11}$$

where $r_0(x) = r(x, 0)$, $s_0(x) = s(x, 0)$; clearly, (2.11) has no relevance *vis á vis* the system (2.9) for which $\alpha = 0$, and thus Hattori's nonexistence results cannot be carried over to establish singularity development in smooth solutions of the inhomogeneous quasilinear hyperbolic system (2.3) which is of interest in this paper.

In order to establish that (2.3) may be rewritten in the form (2.9), for appropriately defined r, s, and Φ we introduce the functions

$$r(x, t) = -\frac{1}{\mu_0} B(x, t) + \int_0^{D(x,t)} \sqrt{\frac{\mathscr{E}'(\zeta)}{\mu_0}}\, d\zeta,$$
$$s(x, t) = -\frac{1}{\mu_0} B(x, t) - \int_0^{D(x,t)} \sqrt{\frac{\mathscr{E}'(\zeta)}{\mu_0}}\, d\zeta, \tag{2.12}$$

and set

$$\lambda = -\sqrt{\mathscr{E}'(D(x, t))/\mu_0}, \quad \nu = \sqrt{\mathscr{E}'(D(x, t))/\mu_0}; \tag{2.13a}$$

$${}' = \frac{\partial}{\partial t} + \lambda(x, t)\frac{\partial}{\partial x}, \qquad {}` = \frac{\partial}{\partial t} + \nu(x, t)\frac{\partial}{\partial x}. \tag{2.13b}$$

By (2.12) and (2.13a),

$$\frac{\partial r}{\partial t}(x, t) = -\frac{1}{\mu_0}\frac{\partial B}{\partial t}(x, t) - \lambda(x, t)\frac{\partial D}{\partial t}(x, t), \tag{2.14a}$$

$$\frac{\partial r}{\partial x}(x, t) = -\frac{1}{\mu_0}\frac{\partial B}{\partial x}(x, t) - \lambda(x, t)\frac{\partial D}{\partial x}(x, t), \tag{2.14b}$$

with similar results for $(\partial s/\partial t)(x, t)$ and $(\partial s/\partial x)(x, t)$.

Combining the equations in (2.3) and using (2.14a,b) we find that

$$\frac{\partial r}{\partial t}(x, t) + \lambda(x, t)\frac{\partial r}{\partial x}(x, t) = -\sqrt{\frac{\mathscr{E}'(D(x, t))}{\mu_0}}\,\Sigma(D(x, t)) \tag{2.15}$$

or

$$r'(x, t) = -\sqrt{\frac{\mathscr{E}'(D(x, t))}{\mu_0}}\,\Sigma(D(x, t)). \tag{2.16}$$

In a similar manner we obtain from (2.3) and the expressions for $\partial s/\partial t$, $\partial s/\partial x$, which are analogous to (2.14a,b),

$$s`(x, t) = +\sqrt{\frac{\mathscr{E}'(D(x, t))}{\mu_0}}\,\Sigma(D(x, t)). \tag{2.17}$$

Now, by using (2.12),

$$r(x, t) - s(x, t) = 2\int_0^{D(x,t)} \sqrt{\frac{\mathscr{E}'(\zeta)}{\mu_0}}\, d\zeta \equiv \eta(x, t).$$

As $\eta = \hat{\eta}(D(x, t))$ satisfies

$$\frac{d\hat{\eta}}{dD} = 2\sqrt{\frac{\mathscr{E}'(D)}{\mu_0}} > 0,$$

$\hat{\eta}$ is monotonic in D and we may define an inverse $\hat{\eta}^{-1}$ such that $D(x, t) = \hat{\eta}^{-1}(r(x, t) - s(x, t))$; thus

$$\lambda(x, t) = -\sqrt{\mathscr{E}'(\hat{\eta}^{-1}(r(x, t) - s(x, t)))/\mu_0} \equiv \hat{\lambda}(r(x, t) - s(x, t)). \tag{2.18}$$

As $\nu(x, t) = -\lambda(x, t)$, $\hat{\nu}(= -\hat{\lambda})$ such that $\nu(x, t) = \hat{\nu}(r(x, t) - s(x, t))$. Finally, if we define

$$\Phi(\kappa) = -\sqrt{\frac{\mathscr{E}'(\hat{\eta}^{-1}(\kappa))}{\mu_0}}\,\Sigma(\hat{\eta}^{-1}(\kappa)), \quad \kappa \in R^1. \tag{2.19}$$

then Eqs. (2.16), (2.17) assume the form

$$\begin{aligned} r'(x, t) &= \Phi(r(x, t) - s(x, t)), \\ s'(x, t) &= -\Phi(r(x, t) - s(x, t)). \end{aligned} \tag{2.20}$$

The functions $r(x, t)$, $s(x, t)$ are, of course, the Riemann invariants naturally associated with our system (2.5) and along the characteristic curves $x_1(t; \beta_1)$, $x_2(t; \beta_2)$, defined by the solutions of the initial-value problems

$$\begin{aligned} \frac{dx_1}{dt} &= \hat{\lambda}(r(x_1, t) - s(x_1, t)); \; x_1(0, \beta_1) = \beta_1, \\ \frac{dx_2}{dt} &= \hat{\nu}(r(x_2, t) - s(x_2, t)); \; x_2(0, \beta_2) = \beta_2, \end{aligned} \tag{2.21}$$

the system of partial differential equations (2.20) reduces to the pair of ordinary differential equations

$$\begin{aligned} \frac{d}{dt} r(x_1(t, \beta_1), t) &= \Phi(r(x_1(t, \beta_1), t) - s(x_1(t, \beta_1), t)), \\ \frac{d}{dt} s(x_2(t, \beta_2), t) &= -\Phi(r(x_2(t, \beta_2), t) - s(x_2(t, \beta_2), t)). \end{aligned} \tag{2.22}$$

Associated with the system (2.20) we have initial conditions of the form

$$r(x, 0) = r_0(x), \quad s(x, 0) = s_0(x), \tag{2.23}$$

where $r_0(\cdot)$, $s_0(\cdot)$ are assumed to be periodic in x and of class C^1 for $x > 0$. In the standard fashion we now extend $r(\cdot, t)$, $s(\cdot, t)$, $r_0(\cdot)$, and $s_0(\cdot)$ to all of R^1, in such a manner that the extended initial data is periodic and of class C^1 for $x \in R^1$, and we consider the initial-value problem (2.20), (2.23) for $-\infty < x < \infty$, $t > 0$.

3. A PRIORI BOUNDS ON C^1 SOLUTIONS

Up till now we have required only that our constitutive theory conform to the hypothesis that $\forall \zeta \in R^1$, $(\zeta\bar{\epsilon}(\zeta))' > 0$ which, in turn, implies that $\mathscr{E}'(\zeta) > 0$, $\forall \zeta \in R^1$. We now make the further assumption that $\mathscr{E}''(0) > 0$ and that $\mathscr{E}(\cdot)$ and $\bar{\sigma}(\cdot)$ jointly satisfy the condition

$$\sup_{\eta} \left| \frac{d}{d\eta} \left(\sqrt{\mathscr{E}'(\hat{\eta}^{-1}(\eta))}\,\Sigma(\hat{\eta}^{-1}(\eta))\right) \right| = \mathscr{K} < \infty. \tag{3.1}$$

Now, as $\hat{\eta}(0) = 0$, we clearly have

$$\Phi(0) = -\sqrt{\frac{\mathscr{E}'(0)}{\mu_0}}\,\Sigma(0) \equiv -\sqrt{\frac{\mathscr{E}'(0)}{\mu_0}}\,\bar{\sigma}(\mathscr{E}(0))\mathscr{E}(0) = 0. \tag{3.2}$$

so that $\Phi(r - s) = \int_0^{r-s} d/d\eta \Phi(\eta)\, d\eta$. Therefore, in view of (3.1),

$$|\Phi(r - s)| \leq \sup_\eta \left|\frac{d\Phi(\eta)}{d\eta}\right| (|r| + |s|) \leq \frac{\ell}{\sqrt{\mu_0}} (|r| + |s|).$$

For future reference we define $M = 2\ell/\sqrt{\mu_0}$ so that the above estimate assumes the form

$$|\Phi(r - s)| \leq \frac{M}{2} (|r| + |s|). \tag{3.3}$$

The last estimate is the basis for the following.

LEMMA

Let (r, s) be a C^1 solution of the initial-value problem (2.20), (2.23) for $0 \leq t \leq T$, $T < \infty$. Then for all $t \leq T$, and $x \in (-\infty, \infty)$,

$$|r(x, t)| + |s(x, t)| \leq (|r_0| + |s_0|)e^{MT}, \tag{3.4}$$

where $|r_0| = \sup_x |r_0(x)|$, $|s_0| = \sup_x |s_0(x)|$.

Proof. We consider the characteristic curves which are defined by the solutions of the initial-value problems (2.21), denoting the solutions of these problems, respectively, by $x_1(t; \beta_1)$ and $x_2(t; \beta_2)$; along these curves the system (2.20) reduces to the set of nonlinear ordinary differential equations (2.22). By integrating Eqs. (2.22) along their respective characteristics we then obtain

$$r(x_1, t) = r_0(\beta_1) + \int_0^t \Phi(r(x_1, \tau) - s(x_1, \tau))\, d\tau, \tag{3.5a}$$

$$s(x_2, t) = s_0(\beta_2) - \int_0^t \Phi(r(x_2, \tau) - s(x_2, \tau))\, d\tau. \tag{3.5b}$$

Thus, for $t \leq T$,

$$|r(x_1, t)| \leq |r_0(\beta_1)| + \int_0^t |\Phi(r(x_1, \tau) - s(x_1, \tau))|\, d\tau, \tag{3.6a}$$

$$|s(x_2, t)| \leq |s_0(\beta_1)| + \int_0^t |\Phi(r(x_2, \tau) - s(x_2, \tau))|\, d\tau. \tag{3.6b}$$

But, in view of (3.3),

$$|\Phi(r(x_1, \tau) - s(x_1, \tau))| \leq \frac{M}{2} (|r(x_1, \tau)| + |s(x_1, \tau)|),$$

$$|\Phi(r(x_2, \tau) - s(x_2, \tau))| \leq \frac{M}{2} (|r(x_2, \tau)| + |s(x_2, \tau)|);$$

and therefore, (3.6a,b) imply that for $t \leq T$,

$$|r(x_1, t)| \leq |r_0(\beta_1)| + \frac{M}{2} \int_0^t |r(x_1, \tau)|\, d\tau + \frac{M}{2} \int_0^t |s(x_1, \tau)|\, d\tau, \tag{3.7a}$$

$$|s(x_2, t)| \leq |s_0(\beta_2)| + \frac{M}{2} \int_0^t |s(x_1, \tau)|\, d\tau + \frac{M}{2} \int_0^t |r(x_2, \tau)|\, d\tau. \tag{3.7b}$$

If we set $R(t) = \sup_x|r(x, t)|$, $S(t) = \sup_x|s(x, t)|$, then

$$|r(x_1, t)| \le |r_0| + \frac{M}{2}\int_0^t (R(\tau) + S(\tau))\, d\tau, \tag{3.8a}$$

$$|s(x_2, t)| \le |s_0| + \frac{M}{2}\int_0^t (R(\tau) + S(\tau))\, d\tau. \tag{3.8b}$$

Because of the assumed periodicity of the initial data, $r(x, t)$ and $s(x, t)$ are periodic in x for each $t \le T$; thus for $t = \hat{t}$, $\hat{x}_1$, $\hat{x}_2$ such that $R(\hat{t}) = |r(\hat{x}_1, \hat{t})|$, $S(\hat{t}) = |s(\hat{x}_2, \hat{t})|$. We now choose $\omega_1 = \omega_1(t)$, $\omega_2 = \omega_2(t)$ such that $\hat{x}_1 = x_1(\omega_1(\hat{t}), \hat{t})$, $\hat{x}_2 = x_2(\omega_2(\hat{t}), \hat{t})$. Then for each $t = \hat{t}$, $R(\hat{t}) = |r(x_1(\omega_1(\hat{t}), \hat{t}), \hat{t})|$ and $S(\hat{t}) = |s(x_2(\omega_2(\hat{t}), \hat{t}), \hat{t})|$. Thus, by choosing $x_1 = x_2(\omega_1(t), t)$, $x_2 = x_2(\omega_2(t), t))$ for each $t \le T$, on the left-hand sides of (3.7a,b), respectively, we obtain from these estimates the bounds

$$R(t) \le |r_0| + \frac{M}{2}\int_0^t (R(\tau) + S(\tau))\, d\tau, \tag{3.9a}$$

$$S(t) \le |s_0| + \frac{M}{2}\int_0^t (R(\tau) + S(\tau))\, d\tau. \tag{3.9b}$$

By defining $W(t) = R(t) + S(t)$ and adding (3.9a,b) we then obtain the estimate

$$W(t) \le |r_0| + |s_0| + M\int_0^t W(\tau)\, d\tau. \tag{3.10}$$

This last result implies (by Gronwall's inequality) that

$$W(t) \le (|r_0| + |s_0|)e^{Mt}, \quad t \le T \tag{3.11}$$

But, for each $t \le T$, $x \in (-\infty, \infty)$, $W(t) \ge (|r(x, t)| + |s(x, t)|)$ by virtue of the definitions of $R(\cdot)$, $S(\cdot)$, and $W(\cdot)$, and (3.4) follows. ■

4. SINGULARITY FORMATION

In this section we begin by assuming that there exists a C^1 solution $(r(x, t), s(x, t))$ of the initial-value problem (2.20), (2.23) for $0 \le t \le T$, $T > 0$ arbitrary; we want to derive a contradiction to this assumption by proving that for $|r_0|$, $|s_0|$ sufficiently small, and $(\partial r/\partial x)(x, 0)$ sufficiently large, for some x, $(\partial r/\partial x)(x, t) \to \infty$ as $t \to t_\infty < \infty$. To this end we first rewrite the first equation in the set (2.20) in the form

$$\frac{\partial r}{\partial t} - \hat{\nu}(r - s)\frac{\partial r}{\partial x} = \Phi(r - s), \tag{4.1}$$

where

$$\hat{\nu}(r - s) \equiv \sqrt{\frac{\mathscr{E}'(\hat{\eta}^{-1}(r - s))}{\mu_0}},$$

and $\Phi(r - s) = -\nu(r - s)\,\Sigma(\hat{\eta}^{-1}(r - s)) \equiv \lambda(r - s)\,\Sigma(\hat{\eta}^{-1}(r - s))$. Differentiating (4.1) through with respect to x, we obtain

$$\frac{\partial^2 r}{\partial t \partial x} - \hat{\nu}\frac{\partial^2 r}{\partial x^2} - \frac{\partial \hat{\nu}}{\partial x} = \Phi'(r - s)\left(\frac{\partial r}{\partial x} - \frac{\partial s}{\partial x}\right). \tag{4.2}$$

where

$$\Phi'(r - s) = \left.\frac{d\Phi(\eta)}{d\eta}\right|_{\eta = r - s}.$$

As $' = \partial/\partial t - \hat{v}\partial/\partial x$ we may put (4.2) in the form

$$\left(\frac{\partial r}{\partial x}\right)' + \frac{\partial \hat{v}}{\partial x}\frac{\partial r}{\partial x} = \Phi'(r - s)\left(\frac{\partial r}{\partial x} - \frac{\partial s}{\partial x}\right). \tag{4.3}$$

But

$$\frac{\partial \hat{v}}{\partial x} = \frac{\partial \hat{v}}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \hat{v}}{\partial s}\frac{\partial s}{\partial x} \equiv \frac{\partial \hat{v}}{\partial r}\left(\frac{\partial r}{\partial x} - \frac{\partial s}{\partial x}\right)$$

and thus if we set $\Psi(r - s) = \Phi'(r - s)$,

$$\left(\frac{\partial r}{\partial x}\right)' - \frac{\partial \hat{v}}{\partial r}\left(\frac{\partial r}{\partial x} - \frac{\partial s}{\partial x}\right)\frac{\partial r}{\partial x} = \Psi(r - s)\left(\frac{\partial r}{\partial x} - \frac{\partial s}{\partial x}\right) \tag{4.4}$$

or

$$\left(\frac{\partial r}{\partial x}\right)' + \frac{\partial \hat{v}}{\partial r}\frac{\partial s}{\partial x}\frac{\partial r}{\partial x} = \frac{\partial \hat{v}}{\partial r}\left(\frac{\partial r}{\partial x}\right)^2 + \Psi(r - s)\left(\frac{\partial r}{\partial x} - \frac{\partial s}{\partial x}\right). \tag{4.5}$$

By combining Eqs. (2.20) we have $r' - s\grave{} = 2\Phi$. However, $s' = \partial s/\partial t - \hat{v}\partial s/\partial x$, and, thus, as $s\grave{} = \partial s/\partial t + \hat{v}\partial s/\partial x$ it follows that

$$\frac{\partial s}{\partial x} = -\frac{\Phi}{\hat{v}} + \frac{(r' - s')}{2\hat{v}}.$$

Substituting this last result into (4.5) and simplifying we obtain

$$\left(\frac{\partial r}{\partial x}\right)' + \frac{\partial \hat{v}}{\partial r}\cdot\frac{(r' - s')}{2\hat{v}}\frac{\partial r}{\partial x} - \frac{\Phi}{\hat{v}}\frac{\partial \hat{v}}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial \hat{v}}{\partial r}\left(\frac{\partial r}{\partial x}\right)^2 + \Psi\frac{\partial r}{\partial x} - \Psi\left(\frac{r' - s'}{2\hat{v}}\right) + \frac{\Phi\Psi}{\hat{v}}. \tag{4.6}$$

We now note that

$$(\log \hat{v})' = \frac{\hat{v}'}{\hat{v}} = \frac{(\partial\hat{v}/\partial r)r' + (\partial\hat{v}/\partial s)s'}{\hat{v}},$$

or

$$(\log \hat{v})' = \frac{\partial\hat{v}/\partial r}{\hat{v}}(r' - s');$$

using this fact we may rewrite (4.6) in the form

$$\left(\frac{\partial r}{\partial x}\right)' + \frac{1}{2}(\log \hat{v})'\frac{\partial r}{\partial x} = \frac{\partial \hat{v}}{\partial r}\left(\frac{\partial r}{\partial x}\right)^2 + \left(\Psi + \frac{\Phi}{\hat{v}}\right)\frac{\partial r}{\partial x} - \Psi\left(\frac{r' - s'}{2\hat{v}}\right) + \frac{\Phi}{\hat{v}}\Psi. \tag{4.7}$$

Multiplying the last equation through by $\hat{\nu}^{1/2}$ and using the fact that $\hat{\nu}^{1/2}(\log \hat{\nu})' = \hat{\nu}^{-1/2}\hat{\nu}'$ we obtain from (4.7)

$$\left(\nu^{1/2}\frac{\partial r}{\partial x}\right)' = \hat{\nu}^{1/2}\frac{\partial \hat{\nu}}{\partial r}\left(\frac{\partial r}{\partial x}\right)^2 + \left(\Psi + \frac{\Phi}{\hat{\nu}}\frac{\partial \hat{\nu}}{\partial r}\right)\hat{\nu}^{1/2}\frac{\partial r}{\partial x} - \frac{\Psi}{2}\hat{\nu}^{-1/2}(r' - s') + \hat{\nu}^{-1/2}\Phi\Psi. \quad (4.8)$$

We now set $\chi = \hat{\nu}^{1/2}\partial r/\partial x$, thus reducing (4.8) to

$$\chi' = \hat{\nu}^{-1/2}\frac{\partial \hat{\nu}}{\partial r}\chi^2 + \left(\Psi + \frac{\Phi}{\hat{\nu}}\frac{\partial \nu}{\partial r}\right)\chi - \frac{\Psi}{2}\hat{\nu}^{-1/2}(r' - s') + \hat{\nu}^{-1/2}\Phi\Psi. \quad (4.9)$$

At this stage of the analysis it is natural to introduce the definition

$$\mathscr{F}(r - s) = -\frac{1}{2}\int_0^{r-s}\Psi(\zeta)\hat{\nu}^{-1/2}(\zeta)\,\mathrm{d}\zeta, \quad (4.10)$$

so that (4.9) becomes

$$\chi' = \hat{\nu}^{-1/2}\frac{\partial \nu}{\partial r}\chi^2 + \left(\Psi + \frac{\Phi}{\hat{\nu}}\frac{\partial \hat{\nu}}{\partial r}\right)\chi + \hat{\nu}^{-1/2}\Phi\Psi + \mathscr{F}'. \quad (4.11)$$

In order to simplify the subsequent analysis we now introduce the notation

$$\phi = \Psi + \frac{\Phi}{\hat{\nu}}\frac{\partial \hat{\nu}}{\partial r}; \quad \psi = \hat{\nu}^{-1/2}\Phi\Psi. \quad (4.12)$$

We note that both ϕ, ψ are functions of $r - s$. In view of the definitions (4.12), (4.11) may be rewritten as

$$\chi' = \hat{\nu}^{-1/2}\frac{\partial \hat{\nu}}{\partial r}\chi^2 + \phi\chi + \psi + \mathscr{F}'. \quad (4.13)$$

We make the natural definition $\theta \equiv \chi - \mathscr{F}$ and (4.13) becomes

$$\theta' = \hat{\nu}^{-1/2}\frac{\partial \hat{\nu}}{\partial r}\theta^2 + \mathscr{H}\theta + \mathscr{G}, \quad (4.14)$$

where

$$\mathscr{H} \equiv 2\nu^{-1/2}\frac{\partial \hat{\nu}}{\partial r}\mathscr{F} + \phi \quad \text{and} \quad \mathscr{G} \equiv \hat{\nu}^{-1/2}\frac{\partial \hat{\nu}}{\partial r}\mathscr{F}^2 + \phi\mathscr{F} + \psi$$

are both functions of $r - s$. We now pause to examine the coefficient $\hat{\nu}^{-1/2}\partial\hat{\nu}/\partial r$ of the quadratic term in (4.14); we compute directly from the relations

$$r - s \equiv \eta = \hat{\eta}(D) \equiv 2\int_0^D \sqrt{\frac{\mathscr{E}'(\zeta)}{\mu_0}}\,\mathrm{d}\zeta$$

and

$$\hat{\nu}(r - s) = \sqrt{\frac{\mathscr{E}'(\hat{\eta}^{-1}(r - s))}{\mu_0}} \equiv \sqrt{\frac{\mathscr{E}'(D)}{\mu_0}}.$$

that

$$\frac{\partial D}{\partial r} = \frac{\partial D}{\partial \eta} = \frac{1}{\dfrac{\partial \eta}{\partial D}} = \frac{\sqrt{\mu_0}}{2} \cdot \frac{1}{\sqrt{\mathcal{E}'(D)}} \quad \text{or} \quad \frac{\partial D}{\partial r} = \frac{\sqrt{\mu_0}}{2} \frac{1}{\sqrt{\mathcal{E}'(\hat{\eta}^{-1}(r-s))}}.$$

Also,

$$\frac{\partial \hat{\nu}}{\partial r} = \frac{\partial \hat{\nu}}{\partial D} \cdot \frac{\partial D}{\partial r}$$

so $\partial\hat{\nu}/\partial r \equiv \frac{1}{4}(\mathcal{E}'(D))^{-1}\mathcal{E}''(D)$, where $D = \hat{\eta}^{-1}(r - s)$. Therefore

$$\hat{\nu}^{-1/2}\frac{\partial \hat{\nu}}{\partial r} = \frac{1}{4}\mu_0^{1/4}\frac{\mathcal{E}''(D)}{(\mathcal{E}'(D))^{5/4}}; \quad D = \hat{\eta}^{-1}(r-s). \tag{4.15}$$

In view of our assumption that $\mathcal{E}''(0) > 0$ it follows that $\exists \Lambda > 0$ such that $\mathcal{E}''(\zeta) > 0$ for $|\zeta| \leq \Lambda$. But for $t \leq T$, $|r - s| \leq (|r_0| + |s_0|)e^{MT} \leq \Lambda$ provided we choose $r_0(\cdot)$, $s_0(\cdot)$ such that $|r_0| + |s_0| \leq \Lambda e^{-MT}$; with such a choice of the initial data, it follows that for $t \leq T$, $-\infty < x < \infty$ $\mathcal{E}''(\eta^{-1}(r(x, t) - s(x, t))) > 0$ and thus by (4.15) that $\hat{\nu}^{-1/2}\partial\hat{\nu}/\partial r > 0$, $0 \leq t \leq T$. If we define

$$\Gamma = \inf \hat{\nu}^{-1/2}\frac{\partial \hat{\nu}}{\partial r}; \quad h = \sup|\mathcal{H}|, \quad g = \inf \mathcal{G}, \tag{4.16}$$

where the respective inf and sup are, in each case, taken over the bounded set of arguments $\{\eta \equiv r - s | |\eta| \leq \Lambda,\ 0 \leq t \leq T\}$, then Eq. (4.14) implies the ordinary differential inequality

$$\theta' \geq \Gamma\theta^2 + \mathcal{H}\theta + \mathcal{G}. \tag{4.17}$$

However,

$$\mathcal{H}\theta \geq -\frac{\Gamma}{2}\theta^2 - \frac{1}{2\Gamma}\mathcal{H}^2 \geq -\frac{\Gamma}{2}\theta^2 - \frac{1}{2\Gamma}h^2,$$

so for $0 \leq t \leq T$,

$$\theta' \geq \frac{\Gamma}{2}\theta^2 - \frac{1}{2\Gamma}h^2 + g. \tag{4.18}$$

We may, without loss of generality, assume that $g \leq 0$ [if $g > 0$, (4.18) is then strengthened by dropping g]. Thus

$$\theta' \geq \frac{\Gamma}{2}(\theta^2 - y^2), \quad 0 \leq t \leq T; \quad y \equiv \left[\left(\frac{h}{\Gamma}\right)^2 - \frac{2}{\Gamma}g\right]^{1/2}. \tag{4.19}$$

With (4.19) we have associated initial data

$$\begin{aligned}\theta(x, 0) &\equiv \chi(x, 0) - \mathcal{A}(x, 0)\\ &= \hat{\nu}^{1/2}(r_0(x) - s_0(x))\frac{\partial r}{\partial x}(x, 0) + \frac{1}{2}\int_0^{r_0(x) - s_0(x)}\Psi(\zeta)\hat{\nu}^{-1/2}(\zeta)\,d\zeta.\end{aligned} \tag{4.20}$$

The solution of (4.19), (4.20) is now compared with the solution of the initial-value problem

$$\hat{\theta}' = \frac{\Gamma}{2}(\hat{\theta}^2 - y^2), \quad 0 \leq t \leq T; \quad \hat{\theta}(x, 0) = \theta(x, 0). \tag{4.21}$$

By a standard comparison theorem[18], $\theta(x, t) \geq \hat{\theta}(x, t)$ for $0 \leq t \leq T$. However, the solution of (4.21) is easily seen to be given by

$$\frac{1}{\hat{\theta}(x, t) + \mathcal{J}} = \frac{e^{\mathcal{J}\Gamma t}}{\theta(x, 0) + \mathcal{J}} + \frac{1}{2\mathcal{J}}(1 - e^{\mathcal{J}\Gamma t}). \tag{4.22}$$

By (4.2) it follows that $\exists t_\infty < \infty$ such that $\theta(x, t) \to \infty$ as $t \to t_\infty$ iff for such t_∞

$$\lim_{t \to t_\infty} \left[\frac{e^{\mathcal{J}\Gamma t}}{\theta(x, 0) + \mathcal{J}} + \frac{1}{2\mathcal{J}}(1 - e^{\mathcal{J}\Gamma t})\right] = 0. \tag{4.23}$$

But, (4.23) is satisfied iff

$$\frac{2\mathcal{J}}{\theta(x, 0) + \mathcal{J}} = 1 - e^{-\mathcal{J}\Gamma t_\infty} \tag{4.24}$$

for some $t_\infty < \infty$. As $0 < 1 - e^{-\mathcal{J}\Gamma t} < 1$ for all $t > 0$, it follows that (4.24) is satisfied iff $\theta(x, 0)$ may be chosen so as to satisfy

$$0 < \frac{2\mathcal{J}}{\theta(x, 0) + \mathcal{J}} < 1, \tag{4.25}$$

in which case

$$t_\infty = -\frac{1}{\mathcal{J}\Gamma} \ln\left[1 - \frac{2\mathcal{J}}{\theta(x, 0) + \mathcal{J}}\right] > 0. \tag{4.26}$$

However, in view of (4.20), $\theta(x, 0)$ will certainly satisfy (4.25) if at some x, $\partial r/\partial x(x, 0)$ is positive and sufficiently large. Thus, for $\partial r/\partial x(x, 0)$ positive and sufficiently large $\hat{\theta}(x, t)$ (and, then, $\theta(x, t)$ as well) approaches ∞ as $t \to t_\infty$, where $t_\infty > 0$ is given by (4.26). Moreover, t_∞, as given by (4.26), will satisfy $t_\infty < T$ provided $\partial r/\partial x(x, 0) > 0$ is chosen so large that

$$\theta(x, 0) > \frac{2\mathcal{J}}{1 - e^{-\mathcal{J}\Gamma T}} - \mathcal{J} \tag{4.27}$$

But

$$\theta(x, t) = \nu^{1/2}(r(x, t) - s(s, t))\frac{\partial r}{\partial x}(x, t) + \frac{1}{2}\int_0^{r(x,t)-s(x,t)} \Psi(\zeta)\hat{\nu}^{1/2}(\zeta)\, d\zeta, \tag{4.28}$$

and thus $\partial r/\partial x(x, t) \to \infty$ as $t \to t_\infty$, for $\partial r/\partial x(x, 0) > 0$ and sufficiently large at x, where $t_\infty < T$ is given by (4.26); this contradicts our assumption that $r(x, t) \in C^1$ (in (x, t)) for all t, $0 \leq t < \infty$. As

$$\frac{\partial r}{\partial x}(x, t) = -\frac{1}{\mu_0}\frac{\partial B(x, t)}{\partial x} + \sqrt{\frac{\epsilon'(D(x, t))}{\mu_0}}\frac{\partial D}{\partial x}(x, t) \tag{4.29}$$

we have established the following result on singularity formation in the plane-wave–nonlinear dielectric interaction problem.

Theorem

Consider the inhomogeneous quasilinear system (2.3) with periodic initial data $B(x, 0) = B_0(x)$, $D(x, 0) = D_0(x)$. Suppose that $\epsilon'(\zeta) > 0$, $\forall \zeta \in R^1$, $\epsilon''(0) > 0$, and that $\epsilon(\cdot)$, $\bar{\sigma}(\cdot)$ jointly satisfy (3.1). Then a C^1 solution $(B(x, t), D(x, t))$ cannot exist for all $t > 0$ if $\sup_x|B_0(x)|$,

$\sup_x|D_0(x)|$ are chosen sufficiently small while

$$\sqrt{\frac{\mathcal{E}'(D_0(x))}{\mu_0}}\, D_0'(x) - \frac{1}{\mu_0} B_0'(x)$$

is chosen so as to be positive and sufficiently large at some x. In fact, given $T > 0$, $\exists \rho_1(T) > 0$, $\rho_2(T) > 0$ such that for $\max(\sup_x|B_0(x)|, \sup|D_0(x)|) < \rho_2(T)$ and

$$\sqrt{\frac{\mathcal{E}'(D_0(x))}{\mu_0}}\, D_0'(x) - \frac{1}{\mu_0} B_0'(x) > \rho_2(T), \quad \exists t_\infty < T$$

such that

$$\lim_{t\to t_\infty}\left(\left[\frac{\partial B(x,t)}{\partial x}\right]^2 + \left[\frac{\partial D(x,t)}{\partial x}\right]^2\right)^{1/2} = \infty.$$

Remarks

It is clear from the conditions cited in the theorem that singularities also form, for $\sup_x|B_0(x)$ and $\sup_x|D_0(x)|$ chosen sufficiently small, if $D_0'(x) > 0$, $-\infty < x < \infty$, while $B_0'(x) < 0$, $-\infty < x < \infty$, with $|B_0'(x)|$ sufficiently large at some x.

REFERENCES

1. N. Bloembergen, *Nonlinear Optics*. W. A. Benjamin, New York (1965).
2. F. Bloom, Nonexistence of smooth electromagnetic fields in nonlinear dielectrics II: Shock development in a half-space. *J. Math. Anal. Applic.*, (1986) in press.
3. F. DeMartini, C. H. Townes, T. K. Gustafson and P. L. Kelley, Self-steepening of light pulses. *Phys. Rev.* **164**, 312–323 (1967).
4. L. J. F. Broer, Wave propagation in non-linear media. *ZAMP* **16**, 11–26 (1965).
5. A. Jeffrey, Non-dispersive wave propagation in nonlinear dielectrics. *ZAMP* **16**, 741–745 (1968).
6. A. Jeffrey and V. P. Korobeinikov, Formation and decay of electromagnetic shock waves. *ZAMP* **20**, 440–447 (1969).
7. I. G. Kataev, *Electromagnetic Shock Waves*. Iliffe Pub., London (1966).
8. A. Donato and D. Fusco, Some applications of the Riemann method to electromagnetic wave propagation in nonlinear media. *ZAMP* **60**, 537–613 (1980).
9. F. Bloom, Shock formation in an inhomogeneous nonlinear dielectric in the presence of a nonlinear conduction current (in preparation).
10. F. Bloom, Smoothing of plane electromagnetic waves in nonlinear dielectrics (in preparation).
11. P. D. Lax, Development of singularities of solutions in nonlinear hyperbolic partial differential equations. *J. Math. Phys.* **S**, 611–613 (1964).
12. S. Klainerman and A. Majda, Formation of singularities for wave equations including the nonlinear vibrating string. *CPAM* **XXXIII** (1980).
13. T. Nishida, *Nonlinear Hyperbolic Equations and Related Topics in Fluid Dynamics*. Publications Mathematiques D'Orsay 78.02, Universite de Paris-Sud, Department de Mathematique (1978).
14. M. Slemrod, Instability of steady shearing flows in a non-linear viscoelastic fluid. *Arch. Rat. Mech. Anal.* **68**, 211–255 (1978).
15. A. Douglis, Some existence theorems for hyperbolic systems of partial differential equations in two independent variables. *Comm. Pure Appl. Math.* **5**, 119–154 (1952).
16. H. Hattori, Breakdown of smooth solutions in dissipative nonlinear hyperbolic equations. Ph.D. thesis, R.P.I. (1981).
17. Rozhdestviskii, *Translations of the A.M.S.* **101** (1973).
18. V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Vol. 1. Academic Press, New York (1969).